Lecture 6: Rigid Bodies part 2

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3.8 Rectangular components of the moment of a force

Resolving r and **F** into rectangular x, y, and z components simplifies the determination of a force in space (when we aren't provided a known reference point).

$$r = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k}$$
$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$
$$\mathbf{M_O} = M_x\mathbf{i} + M_y\mathbf{j} + M_z\mathbf{k} = \mathrm{r} \times \mathbf{F}$$

and the rectangular components of $\mathbf{M_O}$ are the relations:

$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$

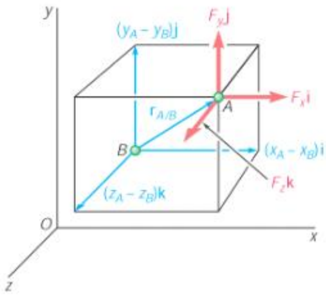


Fig. 3.16

To compute the moment $\mathbf{M_B}$ about an arbitrary point B of a force **F** applied at A, we need to replace the position vector r by a vector from B to A (position vector of A relative to B)

$$\mathrm{r}_{A/B} = \mathrm{r}_A - \mathrm{r}_B$$
$$\mathbf{M_B} = \mathrm{r}_{A/B} \times \mathbf{F} = (\mathrm{r}_A - \mathrm{r}_B) \times \mathbf{F}$$

using the determinant form:

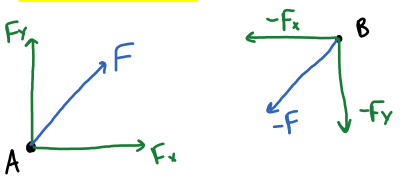
$$\mathbf{M_B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_{A/B} & y_{A/B} & z_{A/B} \\ F_x & F_y & F_z \end{vmatrix} \quad \text{where } x_{\frac{A}{B}} = x_A - x_B, \text{etc.}$$

In 2D problems, assume **F** lies on the xy plane, meaning set z=0 and $\mathrm{F_z} = 0$:

$$\mathbf{M_O} = (\mathrm{x}F_y - yF_x)\mathbf{k}$$
$$M_O = M_z = xF_y - \mathrm{y}F_x$$
$$M_B = (x_A - x_B)F_y - (y_A - y_B)F_x$$

3.12 Moment of a couple

Two forces **F** and **-F** having the same magnitude, parallel lines of action, and opposite sense are said to form a **couple**.

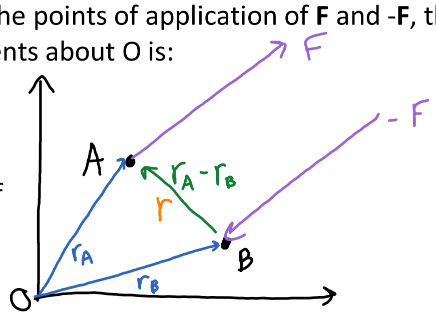


The sum of the two forces in any direction is 0, *however*, the sum of the moments of the two forces about a given point is **not zero**.

- these forces rotate the rigid body, but don't translate it

If $\mathbf{r_A}$ and $\mathbf{r_B}$ are the position vectors of the points of application of **F** and -**F**, then the sum of the moments of the two moments about O is:

$$r_A \times F + r_b \times (-F) = (r_A - r_B) \times F$$



To simplify, we can set $r_A - r_B = r$, and so the sum of the two moments of **F** and **-F** about O is:

$$\mathbf{M} = \mathrm{r} \times \mathbf{F}$$

↳"moment of the couple"

and its magnitude is:

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$M = rF\sin\theta = F\mathrm{d}$, where d is the perpendicular distance between the lines of action of **F** and **-F**. The sense of **M** is defined using the right-hand rule.

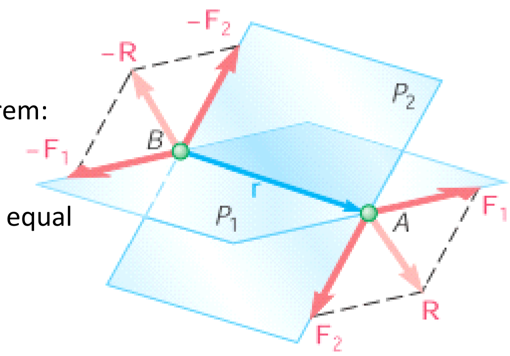
Two couples have equal moments if:
$F_1 d_1 = F_2 d_2$ and they lie on the same plane (or parallel planes) and have the same sense.

3.13 Equivalent couples

When a couple acts on a rigid body, the location of the two forces forming the couple don't matter, or what direction and magnitude they have. The only thing that matters is the **moment of the couple** (magnitude and direction).

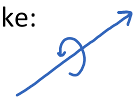
3.14 Addition of couples

$\mathbf{M} = \mathrm{r} \times \mathrm{R} = \mathrm{r} \times (\mathbf{F_1} + \mathbf{F_2})$
which equals, using Varignon's theorem:
$\mathbf{M} = \mathbf{M_1} + \mathbf{M_2} = \mathrm{r} \times \mathbf{F_1} + \mathrm{r} \times \mathbf{F_2}$
The sum of two couples of moments $\mathbf{M}_1$ and $\mathbf{M}_2$ is a couple of moment **M** equal to the vector some of $\mathbf{M_1}$ and $\mathbf{M_2}$.



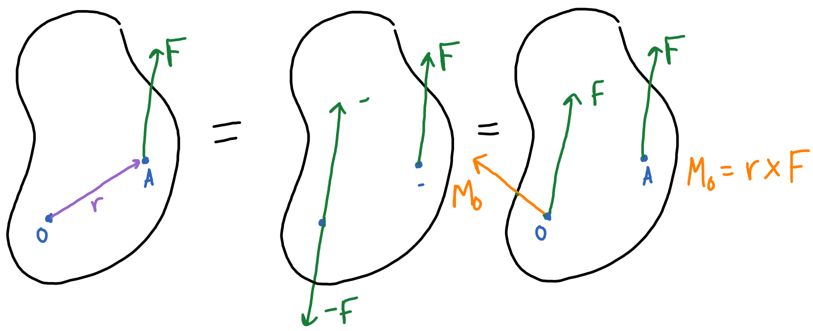
3.15 Couples can be represented by vectors

Couples which have the same moment on the same or parallel planes are **equivalent**. Therefore, we don't need to draw the forces of the couples, only the moment **M** of the couple, and if we have $\mathbf{M}_1$ and $\mathbf{M}_2$ we just need to draw their resultant **M**.

- the vector representing a couple is a **couple vector**, and also a **free vector**
- it looks like:  or: 
- the couple vector **M** can be resolved into its components $\mathbf{M_x}$, $\mathbf{M_y}$, and $\mathbf{M_y}$, representing couples acting on the yz, zx, and xy planes, respectively.

3.16 Resolution of a given force into a force at O and a couple

We have a force **F** acting on a rigid body at A:



We attach 2 forces at point O, **F** and **-F** without affecting the original actions of the rigid body. The force F is now applied at O, and form a couple of moment $\mathbf{M_o}$.
Therefore, any force **F** can be applied at an arbitrary point O provided that a couple is added whose moment is equal to the moment of F about O.
$\mathbf{M_o}$ attached at O and together with **F** are called a **force-couple system**.

We can move **F** in a different point O' and to calculate $\mathbf{M_O}$ we use:
$\mathbf{M_{O'}} = \mathrm{r}' \times \mathbf{F} = (\mathrm{r} + \mathbf{S}) \times \mathbf{F} = \underbrace{\mathrm{r} \times \mathbf{F}}_{M} + \mathbf{S} \times \mathbf{F}$
$\mathbf{M_{O'}} = \mathbf{M_O} + \mathbf{S} \times \mathbf{F}$

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$$\mathbf{M}_{O'} = \mathrm{r}' \times \mathbf{F} = (\mathrm{r} + \mathbf{S}) \times \mathbf{F} = \underbrace{\mathrm{r} \times \mathbf{F}}_{M_O} + \mathbf{S} \times \mathbf{F}$$

$$\mathbf{M}_{O'} = \mathbf{M}_O + \mathbf{S} \times \mathbf{F}$$

Where **S** is the vector joining O' to O. **S** x **F** represents the moment about O' of the force **F** applied at O.

Any force-couple system consisting of force **F** and couple vector $\mathbf{M_O}$ which are mutually perpendicular can be replaced by a single equivalent force.

- this is done by moving **F** in the plane perpendicular to $\mathbf{M_O}$ until its moment about O is equal to the moment of the couple to be eliminated.

3.17 Reduction of a system of forces to one force and one couple

Any system of forces can be reduced to an **equivalent force-couple system** at a given point O.

"moment resultant"

$\mathbf{R} = \Sigma\mathbf{F}$	$\mathbf{M_O^R} = \Sigma(\mathbf{r} \times \mathbf{F})$
obtained by adding all the forces of the system	obtained by adding the moments about O of all the forces of the system

Once the system is reduced to a force and a couple at point O, it can easily be reduced to a force and a couple at another point O':

$$\mathbf{M}_{O'}^{\mathbf{R}} = \mathbf{M_O^R} + \mathbf{S} \times \mathbf{R}$$

Alternatively, we can resolve each position vector **r** and each force **F** into rectangular components:

$$\mathbf{r} = \mathrm{x}\mathbf{i} + \mathrm{y}\mathbf{j} + \mathrm{z}\mathbf{k}$$

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$$

$$\mathbf{M_O^R} = M_x{}^R\mathbf{i} + M_y{}^R\mathbf{j} + M_z{}^R\mathbf{k}$$

3.18 Equivalent system of forces

Two systems of forces are equivalent if they can be reduced to the same force-couple system at a given point O, or:

$\Sigma\mathbf{F} = \Sigma\mathbf{F}'$	$\Sigma\mathbf{M_O} = \Sigma\mathbf{M}'_\mathbf{O}$
$\Sigma\mathbf{F_x} = \Sigma\mathbf{F'}_\mathbf{x}$	$\Sigma\mathbf{M_x} = \Sigma\mathbf{M'}_\mathbf{x}$
$\Sigma\mathbf{F_y} = \Sigma\mathbf{F'}_\mathbf{y}$	$\Sigma\mathbf{M_y} = \Sigma\mathbf{M'}_\mathbf{y}$
$\Sigma\mathbf{F_z} = \Sigma\mathbf{F'}_\mathbf{z}$	$\Sigma\mathbf{M_z} = \Sigma\mathbf{M'}_\mathbf{z}$

The physical significance of these equations is that 2 systems of forces are equivalent if they tend to impart to the rigid body:

- **the same translation** in the x,y,z directions and
- **the same rotation** about the x,y,z axes